

Problem 1. Suppose

- a) $f(x)$ continuous for $x \geq a$.
- b) $f(x) \geq 0$ for $x \geq a$.
- c) $b > a$.

Show that

$$\int_a^\infty f(x)dx \text{ convergent} \iff \int_b^\infty f(x)dx \text{ convergent.}$$

Solution 1.

Lemma 1. For $b > a$,

$$\int_a^\infty f(x)dx = \int_a^b f(x)dx + \int_b^\infty f(x)dx.$$

Proof. By the definition of an indefinite integral,

$$\int_a^\infty f(x)dx = \lim_{n \rightarrow \infty} \int_a^n f(x)dx.$$

We can split this up into $\lim \int_a^b f(x)dx + \lim \int_b^n f(x)dx$ because we now have a definite integral. Again using the definition of an indefinite integral we have

$$\int_a^b f(x)dx + \int_b^\infty f(x)dx = \int_a^\infty f(x)dx.$$

□

Now, since we are trying to show an if and only if relationship, we must prove the implication both ways. First, let $\int_a^\infty f(x)dx$ be convergent. We must show that $\int_b^\infty f(x)dx$ is convergent. By Lemma 1, we have that $\int_a^\infty f(x)dx = \int_a^b f(x)dx + \int_b^\infty f(x)dx$. It is given that $\int_a^\infty f(x)dx$ is convergent. $\int_a^b f(x)dx$ is simply a definite integral, so we know that it has some constant value. With these two convergent integrals, we can define $\int_b^\infty f(x)dx$ in terms of the integrals from a to ∞ and a to b .

$$\int_b^\infty f(x)dx = \int_a^\infty f(x)dx - \int_a^b f(x)dx$$

And so if $\int_a^\infty f(x) dx$ is convergent, then $\int_b^\infty f(x) dx$ must be convergent. \square

Next we must show that the converse is true. Let $\int_b^\infty f(x) dx$ converge. We must show that this implies that $\int_a^\infty f(x) dx$ converges. The argument is very similar to the argument above. Again, by Lemma 1, we have that $\int_a^\infty f(x) dx = \int_a^b f(x) dx + \int_b^\infty f(x) dx$. It is given that $\int_b^\infty f(x) dx$ is convergent. $\int_a^b f(x) dx$ is simply a definite integral, so we know that it has some constant value. So, with Lemma 1, we have $\int_a^\infty f(x) dx$ equal to two integrals with constant value. Therefore, $\int_a^\infty f(x) dx$ must converge. \square

And so the implication is proven in both directions and the if and only if relationship is true. \square

Problem 2. Suppose that $\{a_n\}$ is a sequence such that a_n is defined for $n \geq k$. Suppose that $l > k$. Show

$$\sum_{n=k}^{\infty} a_n \text{ is convergent} \iff \sum_{n=l}^{\infty} a_n \text{ is convergent.}$$

Solution 2.

Lemma 2. For $l > k$,

$$\sum_{n=k}^{\infty} a_n = \sum_{n=k}^l a_n + \sum_{n=l}^{\infty} a_n.$$

Proof. By the definition of an indefinite series,

$$\sum_{n=k}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{n=k}^n a_n.$$

We can split this up into $\lim_{n \rightarrow \infty} \sum_{n=k}^l a_n + \lim_{n \rightarrow \infty} \sum_{n=l}^n a_n$ because we now have a definite series.

Again using the definition of an indefinite series we have

$$\sum_{n=k}^l a_n + \sum_{n=l}^{\infty} a_n = \sum_{n=k}^{\infty} a_n.$$

\square

Now, since we are trying to show an if and only if relationship, we must prove the implication both ways. First, let $\sum_{n=k}^{\infty} a_n$ be convergent. We must show that $\sum_{n=l}^{\infty} a_n$ is convergent. By Lemma 2, we have that $\sum_{n=k}^{\infty} a_n = \sum_{n=k}^l a_n + \sum_{n=l}^{\infty} a_n$. It is given that $\sum_{n=k}^{\infty} a_n$ is convergent. $\sum_{n=k}^l a_n$ is simply a definite series, so we know that it has some constant value.

With these two convergent series, we can define $\sum_{n=l}^{\infty} a_n$ in terms of the series from k to ∞ and k to l .

$$\sum_l^{\infty} a_n = \sum_{n=k}^{\infty} a_n - \sum_{n=k}^l a_n$$

And so if $\sum_{n=k}^{\infty} a_n$ is convergent, then $\sum_{n=l}^{\infty} a_n$ must be convergent. \square

Next we must show that the converse is true. Let $\sum_{n=l}^{\infty} a_n$ converge. We must show that this implies that $\sum_{n=k}^{\infty} a_n$ converges. The argument is very similar to the argument above. Again, by Lemma 2, we have that $\sum_{n=k}^{\infty} a_n = \sum_{n=k}^l a_n + \sum_{n=l}^{\infty} a_n$. It is given that $\sum_{n=l}^{\infty} a_n$ is convergent. $\sum_{n=k}^l a_n$ is simply a definite series, so we know that it has some constant value. So, with Lemma 2, we have $\sum_{n=l}^{\infty} a_n$ equal to two series with constant value. Therefore, $\sum_{n=k}^{\infty} a_n$ must converge. \square

And so the implication is proven in both directions and the if and only if relationship is true.

\square