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**Problem 1.** Suppose

a) $f(x)$ continuous for $x \geq a$.

b) $f(x) \geq 0$ for $x \geq x$.

c) $b > a$.

Show that

$$\int_a^\infty f(x)dx \text{ convergant } \iff \int_b^\infty f(x)dx \text{ convergent.}$$

**Solution 1.**

**Lemma 1.** *For $b > a$,*

$$\int_a^\infty f(x)dx = \int_a^b f(x)dx + \int_b^\infty f(x)dx.$$

*Proof.* By the definition of an indefinite integral,

$$\int_a^\infty f(x)dx = \lim_{n\to\infty} \int_a^n f(x)dx.$$

We can split this up into $\lim \int_a^b f(x)dx + \lim \int_b^n f(x)dx$ because we now have a definite integral. Again using the definition of an indefinite integral we have

$$\int_a^b f(x)dx + \int_b^\infty f(x)dx = \int_a^\infty f(x)dx.$$

□

Now, since we are trying to show an if and only if relationship, we must prove the implication both ways. First, let $\int_a^\infty f(x)dx$ be convergant. We must show that $\int_b^\infty f(x)dx$ is convergant. By Lemma 1, we have that $\int_a^\infty f(x)dx = \int_a^b f(x)dx + \int_b^\infty f(x)dx$. It is given that $\int_a^\infty f(x)dx$ is convergant. $\int_a^b f(x)dx$ is simply a definite integral, so we know that it has some constant value. With these two convergant integrals, we can define $\int_b^\infty f(x)\,dx$ in terms of the integrals from $a$ to $\infty$ and $a$ to $b$.

$$\int_b^\infty f(x)\,dx = \int_a^\infty f(x)\,dx - \int_a^b f(x)\,dx$$

And so if $\int_a^\infty f(x)\,dx$ is convergant, then $\int_b^\infty f(x)\,dx$ must be convergant. $\square$

Next we must show that the converse is true. Let $\int_b^\infty f(x)\,dx$ converge. We must show that this implies that $\int_a^\infty f(x)\,dx$ converges. The argument is very similar to the argument above. Again, by Lemma 1, we have that $\int_a^\infty f(x)dx = \int_a^b f(x)dx + \int_b^\infty f(x)dx$. It is given that $\int_b^\infty f(x)\,dx$ is convergant. $\int_a^b f(x)dx$ is simply a definite integral, so we know that it has some constant value. So, with Lemma 1, we have $\int_b^\infty f(x)\,dx$ equal to two integrals with constant value. Therefore, $\int_b^\infty f(x)\,dx$ must converge. $\square$

And so the implication is proven in both directions and the if and only if relationship is true.

$\square$

**Problem 2.** Suppose that $\{a_n\}$ is a sequence such that $a_n$ is defined for $n \geq k$. Suppose that $l > k$. Show

$$\sum_{n=k}^{\infty} a_n \text{ is convergant} \iff \sum_{n=l}^{\infty} a_n \text{ is convergant.}$$

**Solution 2.**

**Lemma 2.** *For $l > k$,*

$$\sum_{n=k}^{\infty} a_n = \sum_{n=k}^{l} a_n + \sum_{n=l}^{\infty} a_n.$$

*Proof.* By the definition of an indefinite series,

$$\sum_{n=k}^{\infty} a_n = \lim_{n\to\infty} \sum_{n=k}^{n} a_n.$$

We can split this up into $\lim \sum_{n=k}^{l} a_n + \lim \sum_{n=l}^{n} a_n$ because we now have a definite series. Again using the definition of an indefinite series we have

$$\sum_{n=k}^{l} a_n + \sum_{n=l}^{\infty} a_n = \sum_{n=k}^{\infty} a_n.$$

$\square$

Now, since we are trying to show an if and only if relationship, we must prove the implication both ways. First, let $\sum_{n=k}^{\infty} a_n$ be convergant. We must show that $\sum_{n=l}^{\infty} a_n$ is convergant. By Lemma 2, we have that $\sum_{n=k}^{\infty} a_n = \sum_{n=k}^{l} a_n + \sum_{n=l}^{\infty} a_n$. It is given that $\sum_{n=k}^{\infty} a_n$ is convergant. $\sum_{n=k}^{l} a_n$ is simply a definite series, so we know that it has some constant value.

With these two convergant series, we can define $\sum_{n=l}^{\infty} a_n$ in terms of the series from $k$ to $\infty$ and $k$ to $l$.

$$\sum_{l}^{\infty} a_n = \sum_{n=k}^{\infty} a_n - \sum_{n=k}^{l} a_n$$

And so if $\sum_{n=k}^{\infty} a_n$ is convergant, then $\sum_{n=l}^{\infty} a_n$ must be convergant. $\square$

Next we must show that the converse is true. Let $\sum_{n=l}^{\infty} a_n$ converge. We must show that this implies that $\sum_{n=k}^{\infty} a_n$ converges. The argument is very similar to the argument above. Again, by Lemma 2, we have that $\sum_{n=k}^{\infty} a_n = \sum_{n=k}^{l} a_n + \sum_{n=l}^{\infty} a_n$. It is given that $\sum_{n=l}^{\infty} a_n$ is convergant. $\sum_{n=k}^{l} a_n$ is simply a definite series, so we know that it has some constant value. So, with Lemma 2, we have $\sum_{n=l}^{\infty} a_n$ equal to two series with constant value. Therefore, $\sum_{n=l}^{\infty} a_n$ must converge. $\square$

And so the implication is proven in both directions and the if and only if relationship is true. $\square$