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January 5, 2006

**Exercise 3.7 (Euler's constant).** We know from Exercise 6.3 of Chapter 3 (and also the Integral Test) that

$$1 + \frac{1}{2} + \cdots \frac{1}{n} + \cdots = \infty$$

This mean that

$$\lim S_n = \infty,$$

where

$$S_n = 1 + \frac{1}{2} + \cdots \frac{1}{n}.$$

What about the sequence $\{T_n\}$, where

$$\begin{aligned} T_n &= 1 + \frac{1}{2} + \cdots \frac{1}{n} - \ln n \\ &= S_n - \ln n\,? \end{aligned}$$

Calculating a few values of $\{T_n\}$, we have $T_1 = 0$, $T_2 = 0.3069$, $T_3 = 0.4014$, $T_4 = 0.447$, $T_5 = 0.4739$, and $T_{1000} = 0.5767$. Looking at these values, we hypothesize that $\{T_n\}$ is bounded, monotonic, and so convergant. We shall now prove this.

Recall from the proof of Lemma 3.1, that if $f$ is strictly decreasing and continuous on the interval $[1, \infty)$, then for $n \geq 2$,

$$f(2) + f(3) + \cdots f(n) \leq \int_1^n f(x)\,dx \leq f(1) + f(2) + \cdots + f(n-1).$$

Letting $f(x) = \frac{1}{x}$, we have

$$\frac{1}{2} + \cdots + \frac{1}{n} \leq \ln n \leq 1 + \frac{1}{2} + \frac{1}{n-1}.$$

Using the above equation, we can show that $\{T_n\}$ is bounded. By mulipling the inequality by $-1$ (and thereby flipping the directions of the inequalities) and adding $S_n$ we have

$$\left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right) - \frac{1}{2} - \cdots - \frac{1}{n} \geq S_n - \ln n \geq \left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right) - 1 - \frac{1}{2} - \cdots - \frac{1}{n-1}.$$

This collapses to

$$1 \geq S_n - \ln x \geq \frac{1}{n}.$$

So, for all $n$, $S_n - \ln n \leq 1$. That is, it is bounded.

The next step in proving the convergance of $T_n$ is to show that it is monotonic. We have

$$\begin{aligned} T_{n+1} - T_n &= (1 + \cdots + \frac{1}{n+1} - \ln(n+1)) - (1 + \cdots + \frac{1}{n} - \ln n) \\ &= \frac{1}{n+1} - [\ln(n+1) - \ln n]. \end{aligned}$$

To show that $\{T_n\}$ is monotonic, we show that

$$T_{n+1} - T_n < 0.$$

We can use the above equation to show this. We also use the Mean-Value Theorem.

The Mean-Value Theorem states that if a function $f(x)$ is defined and continous on the interval $[a, b]$ and differentiable on $(a, b)$, then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

For this problem, let $f(a) = \ln x$. This function is defined, continous, and differentiable on $[1, \infty]$, which is the interval we are interested in. Let $a = n$ and $b = n + 1$. Then

$$\ln(n+1) - \ln n = \frac{1}{c} \text{ for some } c \text{ such that } n < c < n + 1.$$

We can substitute this into our inequality to get

$$T_{n+1} - \frac{1}{c} < 0 \text{ for } n < c < n + 1.$$

So the left side can range from $\frac{-1}{n^2+n}$ to 0. Therefore $T_{n+1} - T_n < 0$ for all $n$. We have shown that $\{T_n\}$ is bounded and monotonic, therefore it is convergant. $\square$