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Problem. Look at the infinite series

$$1 - \frac{(3/2)^2}{2!} + \frac{(3/2)^4}{4!} - \frac{(3/2)^6}{6!} + \cdots + \frac{(-1)^n (3/2)^{2n}}{(2n)!} \qquad (*)$$

Start by useing the Alternating-Series Test to estimate the value of this infinite series with error less than 0.001. Finally, use your calculator to compute the value of $cos(3/2)$ correct to a fairly large number of decimal places. Come up with a conjecture about the value of the infinite series.

Solution. First we use the Alternating-Series Test to show that (*) is convergent. The Alternating-Series Test states that for $a_n > 0$ for all n, and $\{a_n\}$ is a strictly decreasing sequence, and $a_n \to 0$. Then the infinite series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ is convergent.

Our series can be put into the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ where $a_n = \frac{(3/2)^{2n}}{(2n)!}$. So first we must show that $a_n > 0$ for all n. We see that all numbers involved are positive and exponentiation and factorial do not result in negative numbers for all $n > 1$ which is the n we are interested in, so $a_n > 0$ for all n.

Next we show that the sequence is strictly decreasing, that is, $a_n > a_{n+1}$ for all n. We have

$$a_n > \frac{(3/2)^{2n+2}}{(2n+2)!} = a_n \cdot \frac{(3/2)^2}{(2n+2)(2n+1)}.$$

In order for this to be true, it must be the case that $\frac{(3/2)^2}{(2n+2)(2n+1)} < 1$. So,

$$1 > \frac{(3/2)^2}{(2n+2)(2n+1)} = \frac{9/4}{4n^2+6n+2} = \frac{9}{16n^2+24n+8}.$$

This is true if and only if $16n^2 + 24n > 1$, which it certainly is for $n > 1$. Now, we have that a_n is strictly decreasing.

Finally, for the Alternating-Series Test, we show that $a_n \to 0$. We approach this limit using a somewhat roundabout method. We show that $\sum a_n$ converges, which implies that $a_n \to 0$ by the n-th term test. We use the the Ratio Test. We already have that $a_n > 0$. Then,

$$\lim \frac{a_{n+1}}{a_n} = \frac{\frac{(3/2)^{2n+2}}{(2n+2)!}}{\frac{(3/2)^{2n}}{(2n)!}} = \lim \frac{(3/2)^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(3/2)^{2n}} = \lim \frac{(3/2)^2}{(n+1)(n+2)} = 0.$$

The limit of the ratio is less than 1, so by the Ratio Test, $\sum_{n=1}^{\infty} a_n$ is convergent. Then, by the contrapositive of the n-th term test, $\lim a_n = 0$.

Having shown that all of the condition of the Alternating-Series Test hold true, we conclude that (*) is convergant. $\square$

We can continue to use the Alternating-Series Test to find an approximation to this infinite series with error less than 0.001. The Alternating-Series Test states: Let $\sum_{n=1}^{\infty}(-1)^{n+1}a_n = L$. Then for all n,

$$|S_n - L| < a_{n+1},$$

Where S_n is the partial sum, $S_n = a_1 - a_2 + \cdots + (-1)^{n+1}a_n$.

To find the approximation, we find an $|a_n| \leq 0.001$. With some calculations, we find that $a_3 \approx 0.0158$ and $a_4 \approx 0.0006356$, so we will use a_4. Now, by the Alternating-Series Test, we have that $|S_4 - L| < a_4$. That is, the difference between the partial sum at 4 and the actual limit is less that 0.00063 is less than 0.001. Computing this sum, we get

$$S_4 = 1 - \frac{(3/2)^2}{2!} + \frac{(3/2)^4}{4!} - \frac{(3/2)^6}{6!} + \frac{(3/2)^8}{8!} \approx 0.070752\ldots$$

Now, computing the value of $\cos(3/2)$ with a calculator, we have $\cos(3/2) \approx 0.07073720$. Noting that that S_4 and the calculation of $\cos(3/2)$ are very similar, at least within an error of 0.001, we conject that our infinite series (*) is equal to $\cos(3/2)$.