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January 5, 2006

Problem. For which real numbers $c \geq 0$ does the recursively defined sequence

$$\begin{aligned} a_1 &= c, \\ a_{n+1} &= \frac{a_n^2 + a_n + 2}{4} \end{aligned}$$

converge? If it converges then what is the limit?

Solution. We shall show that for $0 \leq c < 1$, $c = 1$, and $1 < c < 2$, $\lim a_n = 1$. We will also show that for $c = 2$, $\lim a_n = 2$ and for $c > 2$, $\lim a_n = \infty$. First we will show that if a_n converges, it must converge to either 1 or 2. Then we will show that a_n is increasing for $0 \leq c < 1$ and $c > 2$. Then we will show that a_n is decreasing for $1 < c < 2$. Five cases for c will become evident from these lemmas. With these lemmas and proofs by induction, the problem will be solved.

Lemma 1 (L = 1 or 2). *The limit of a_n must be either 2 or 1.*

Proof. If a_n converges, then as n increases a_n gets closer and closer to some limit L. This is also the case for a_{n+1}, so we can replace them in the definition of the sequence to get

$$L = \frac{L^2 + L + 2}{4} \iff 0 = L^2 - 3L + 2 \iff 0 = (L - 2)(L - 1).$$

Consequently, we see that L must be either 2 or 1. □

Lemma 2 (a_n increasing). *a_n is increasing when $0 \leq c < 1$ and $c > 2$.*

Proof. We prove this by induction.

Basis. Let $P(n)$ be the statement

$$a_n \leq a_{n+1} \text{ for } 0 \leq c < 1 \text{ and } c > 2.$$

By the definition of the sequence, $a_1 = c$ and $a_2 = \frac{c^2+c+2}{4}$. Is $a_1 < a_2$? Algebraicly manipulating the inequality we find

$$c \leq \frac{c^2 + c + 2}{4} \iff 0 \leq c^2 - 3c + 2 \iff 0 \leq (c - 2)(c - 1).$$

So this is true when $0 \leq c < 1$ and $c > 2$.

Induction Step. Suppose that $P(n)$ is true. Then our induction hypothesis is

$$a_n \leq a_{n+1} \text{ for } 0 \leq c < 1 \text{ and } c > 2.$$

We show that $P(n+1)$ is true.

$$a_{n+1} = \frac{a_n^2 + a_n + 2}{4} \leq \frac{a_{n+1}^2 + a_{n+1} + 2}{4} = a_{n+2}.$$

This is true by the induction hypothesis and so $P(n+1)$ is true. □

Lemma 3 (a_n decreasing). *a_n is decreasing when $1 < c < 2$.*

Proof. We prove this by induction. Note that it is *very* similar to Lemma 2. We include it for completeness.

Basis. Let $Q(n)$ be the statement

$$a_n \geq a_{n+1} \text{ for } 1 < c < 2.$$

By the definition of the sequence, $a_1 = c$ and $a_2 = \frac{c^2+c+2}{4}$. Is $a_1 < a_2$? Manipulating the inequality we find

$$c \geq \frac{c^2 + c + 2}{4} \iff 0 \geq c^2 - 3c + 2 \iff 0 \geq (c-2)(c-1).$$

So this is true when $0 < c < 1$.

Induction Step. Suppose that $Q(n)$ is true. Then our induction hypothesis is

$$a_n \geq a_{n+1} \text{ for } 1 < c < 2.$$

We show that $Q(n+1)$ is true.

$$a_{n+1} = \frac{a_n^2 + a_n + 2}{4} \geq \frac{a_{n+1}^2 + a_{n+1} + 2}{4} = a_{n+2}.$$

This is true by the induction hypothesis and so $Q(n+1)$ is true. □

It may not be obvious, but the heart of the argument has now been proven in these three lemmas. From lemma 1 we know what a_n converges to, if it converges. From lemmas 2 and

3, we know at what starting points c, a_n is increasing and decreasing. From these ranges, and the addition of the $c = 1$ and $c = 2$, we can cover the entire range of c. There are five cases:

Case 1: ($0 \leq c < 1$). By Lemma 2, note that in this range, a_n is increasing. If we can show that it is bounded, then we know that is must be convergant. Since we know that by Lemma 1, the possible limits for the sequence are 1 and 2, let us guess that when $0 \leq c < 1$, $a_n < 1$ for all n. We will show this by induction.

Basis. Let $R(n)$ be the statement that

$$a_n < 1 \text{ for } 0 \leq c < 1.$$

Then $R(1)$ is $c = a_1 < 1$, which is true by the conditions of this case.

Induction Step. Note that $R(n)$ is our induction hypothesis. Now we show that $R(n+1)$ is true.

$$a_{n+1} = \frac{a_n^2 + a_n + 2}{4} < \frac{1+1+2}{4} = 1$$

This is true, that is, $R(n+1)$ is true, so 1 is an upper bound of a_n. By Lemma 1, the possible limits of a_n are 1 and 2. We have just shown that when $0 \leq c < 1$, a_n is bounded above. By Lemma 2, we know that the sequence is increasing. Consequently, the limit of a_n when $0 \leq c < 1$ is 1.

Case 2: ($c = 1$). Looking at a few terms, $a_1 = 1, a_2 = 1, a_3 = 1$. Let us guess that $a_n = 1$ for all n.

Basis. Let $S(n)$ be the statement that

$$a_n = 1 \text{ for } c = 1.$$

Then $S(1)$ is true, since $a_1 = 1$.

Induction Step. Note that $S(n)$ is our induction hypothesis. We now show that $S(n+1)$ is true. That is,

$$a_{n+1} = \frac{a_n^2 + a_n + 2}{4} = \frac{1^2+1+2}{4} = 1 \qquad \text{(by the induction hypothesis)}$$

Thus $S(n+1)$ is true. Consequently, the limit of a_n when $c = 1$ is 1.

Case 3: $(1 < c < 2)$. By Lemma 3, note that in this range, a_n is decreasing. If we can show that it is bounded, then we know that is must be convergant. Since we know that by Lemma 1, the possible limits for the sequence are 1 and 2, let us guess that when $1 < c < 2$, $a_n > 1$ for all n. We will show this by induction.

Basis. Let $T(n)$ be the statement that

$$a_n > 1 \text{ for } 1 < c < 2.$$

Then $T(1)$ is $c = a_1 > 1$, which is true by the conditions of this case.

Induction Step. Note that $T(n)$ is our induction hypothesis. Now we show that $T(n+1)$ is true.

$$a_{n+1} = \frac{a_n^2 + a_n + 2}{4} > \frac{1+1+2}{4} = 1$$

This is true, that is, $T(n+1)$ is true, so 1 is lower bound of a_n. By Lemma 1, the possible limits of a_n are 1 and 2. We have just shown that when $1 < c < 2$, a_n is bounded below. By Lemma 3, we know that the sequence is decreasing. Consequently, the limit of a_n when $1 < c < 2$ is 1.

Case 4: $(c = 2)$. Looking at a few terms, $a_1 = 2, a_2 = 2, a_3 = 2$. Let us guess that $a_n = 2$ for all n.

Basis. Let $U(n)$ be the statement that

$$a_n = 2 \text{ for } c = 2.$$

Then $S(1)$ is true, since $a_1 = 2$.

Induction Step. Note that $S(n)$ is our induction hypothesis. We now show that $S(n+1)$ is true. That is,

$$a_{n+1} = \frac{a_n^2 + a_n + 2}{4} = \frac{2^2 + 2 + 2}{4} = 2 \qquad \text{(by the induction hypothesis)}$$

Thus $S(n+1)$ is true. Consequently, the limit of a_n when $c = 2$ is 2.

Case 5: $(c > 2)$. By Lemma 2, we know that a_n is increasing on this interval. By Lemma 1, we know that the limit of this sequence must be either 1 or 2. So, since the sequence is increasing away from the only possible limits, it must be divergent.

In conclusion, for $0 \leq c < 1$, $c = 1$, and $1 < c < 2$, $\lim a_n = 1$; for $c = 2$, $\lim a_n = 2$ and for $c > 2$, $\lim a_n = \infty$. $\square$